Problem 10B,2
Suppose T is a bounded operator on a Hilbert space V.

- Prove that $sp(S^{-1}TS) = sp(T)$ for all bounded invertible operator S.
- Prove that $sp(T^*) = \{\bar{\alpha} | \alpha \in sp(T)\}$.
- Prove that if T is invertible, then $sp(T^{-1}) = \{\frac{1}{\alpha} | \alpha \in sp(T)\}$.

Proof.

- $S^{-1}(T - \alpha I)S = S^{-1}TS - \alpha I$.
- $(T - \alpha I)^* = T^* - \bar{\alpha} I$
- $\frac{1}{T} - \alpha I = (I - \alpha T)T^{-1}$.

□

Problem 10B,7
Prove that if T ia an operator on a Hilbert space V satisfying $< Tf, g >=< f, Tg >$ for all $f, g \in V$, then T is a bounded operator.

Proof. Let $f_n, n = 1, 2, \ldots$ be a sequence in V with norm bounded by 1. We need to show norm of Tf_n is also bounded. Consider the operator T_n in V definded by $T_n(g) =< Tf_n, g >, g \in V$. Then $\|T_n\| =\|Tf_n\|$ and for fixed $g \in V$, $|T_n(g)| = | < f_n, Tg > | \leq\|Tg\|$. So by uniform boundedness principle, Tf_n is also bounded. □

Problem 10B,10
Suppose T is a bounded operator on a Hilbert space V such that $< Tf, f >\geq 0$ for all $f \in V$. Prove that $sp(T) \subset [0, \infty)$

Proof. And since $< Tf, f >$ is real, by 10.48 we know T is self-adjoint. Then by 10.49 we know $sp(T) \subset \mathbb{R}$. For any $\alpha < 0$, we have

$$\|Tf - \alpha f\|\|f\| \geq< Tf - \alpha f, f >\geq -\alpha < f, f > .$$

So $T - \alpha I$ has trivial kernel and this is a self adjoint operator, which means $T - \alpha$ is surjective. So $T - \alpha I$ is invertible, which finishes the proof. □

Problem 10B, 23
For T a bounded operator on a Banach space, define e^T by

$$e^T = \sum_{k=0}^{\infty} \frac{T^k}{k!}$$

- Prove the infite sum converges in $B(V)$ and $\|e^T\| \leq e^{\|T\|}$
- Prove that if S, T are bounded operator on a Banach space v such that $ST = TS$, then $e^S e^T = e^{S+T}$.
- Prove that if T is a self-adjoint operators on a complex Hilbert space, then e^{iT} is unitary.

Proof.

- Only need to note that
$$\| \sum_{k=N}^{M} \frac{T^k}{k!} \| \leq \sum_{k=N}^{M} \frac{\|T\|^k}{k!}.$$
- Since $ST = TS$, the proof is the same as the real number case.
- This can be checked easily from definition.

□

Problem 10C,1
Prove that if T is a compact operator on a Hilbert space V and $e_1, e_2, \dots$ is an orthonormal sequence in V, then $\lim_{n\to\infty} Te_n = 0$.

Proof. Argue by contradiction. If $\lim_{n\to\infty} Te_n \neq 0$, passing to a subsequence and multiplying e_n with complex number of norm 1, we can assume
$f = \lim_{n\to\infty} Te_n, \|f\| \geq 1, < Te_n, f >\geq \frac{1}{2}$. Now consider $e = \sum_{n=1}^{\infty} \frac{1}{n} e_n$. Obviously $\|e\| \leq 1000$ but

$$< Te, f >= \infty$$

this gives contradiction. □

Problem 10C,6
Suppose T is a bounded operator on a Hilbert space V. Prove that if there exists an orthonormal basis $\{e_k\}_{k\in\Gamma}$ of V such that

$$\sum_{k\in\Gamma} \|Te_k\|^2 < \infty$$

then T is compact.

Proof. Note by assumption, without loss of generality we can assume Γ is a countable set.
Let $A_k = span\{e_1, e_2, \dots, e_k\}$ and set $T_k = T|_{A_k}$. Then since $range(T_k)$ is finite dimension, it is compact. We claim that $\lim_{n\to\infty} T_n = T$ which shows that T is also compact. In fact, one can easily check that $\|T_k - T\|^2 \leq \sum_{j=k}^{\infty} \|T(e_k)\|^2$ which goes to zero as $k \to \infty$ by assumption. □

Problem 10C,7
Suppose T is a bounded operator on a Hilbert space V. Prove that if $\{e_k\}_{k\in\Gamma}$ and $\{e_k\}_{k\in\Omega}$ are orthonormal nasis of V, then

$$\sum_{k\in\Gamma} \|Te_k\|^2 = \sum_{j\in\Omega} \|Tf_j\|^2$$

Proof. We will show that if the left hand side is finite, the the right hand side is also finite and they are equal. This also implies that if one hand side is infinity, then both side are infinity and the result also holds in this case. By assumption, without loss of generality, we assume Γ is countable infinity set so V is a seperable Hilbert space and also Ω is countable infinity. Assume $f_m = \sum_{n=1}^{\infty} s_{mn} e_n, Te_i = \sum_{j=1}^{\infty} t_{ij} e_j$. Then for any positive integer m,

$$\sum_{n=1}^{\infty} |s_{mn}|^2 = 1.$$

So

$$\sum_{j\in\Omega} \|Tf_j\|^2 = \sum_{j=1}^{\infty} \sum_{i=1}^{\infty} \sum_{k=1}^{\infty} |s_{ij} t_{ik}|^2$$

Since the summation are all positive, we can change the order to get

$$\sum_{j\in\Omega} \|Tf_j\|^2 = \sum_{k=1}^{\infty} \sum_{i=1}^{\infty} \sum_{j=1}^{\infty} |s_{ij} t_{ik}|^2 = \sum_{k=1}^{\infty} \sum_{i=1}^{\infty} |t_{ik}|^2 = \sum_{k\in\Gamma} \|Te_k\|^2.$$

This finishes the proof. □

Problem 10C,10
Suppose T is a surjective but nit injective operator on a vector space V. Show that

$$null(T) \subsetneq null(T^2) \subsetneq null(T^3) \subsetneq \dots$$

Proof. Obviously

$$null(T) \subset null(T^2) \subset null(T^3) \subset \dots$$

We now show that the equality cannot hold. Note that since T is surjective, T^k are all surjective for any positive interger k. So now take k be any positive integer. Let $a, b \in V, a \neq 0$ such that $T(a) = 0, T^k(b) = a$. Thus $b \in null(T^{k+1}), b \notin null(T^k)$. □